Author(s): Neville J Ford; Yubin Yan; Abdul Md Malique

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Numerical treatment of oscillatory functional differential equations

Neville J Ford[*,a], Yubin Yan[1], Md. Abdul Malique[1]

[a]*Department of Mathematics, University of Chester, CH1 4BJ*

Abstract

In this paper we are concerned with *oscillatory* functional differential equations (that is, those equations where all solutions oscillate) under numerical approximation. Our interest is in the preservation of qualitative properties of solutions under numerical discretisation. We give conditions under which an equation is oscillatory, and consider whether the discrete schemes derived using linear $\vartheta-$methods will also be oscillatory. We conclude with some general theory.

Key words:
Functional differential equations, delay equations, oscillatory solutions, numerical schemes, discrete equations.
AMS Subject Classification: 34K11, 39A12, 65Q05

1. Introduction

We are interested in the preservation of fundamental properties of an equation and its solutions under numerical discretisation. This is particularly important when we are investigating functional differential equations which often do not admit an exact closed form solution, and the use of numerical approximation methods becomes essential. It is known that numerical schemes may supress certain properties of an equation, or introduce spurious properties and one needs to be able to select numerical methods that exhibit accurately the important true patterns of behaviour from the original continuous problem. In previous work (see, for example, [10, 11, 12]), one of the authors of the present paper considered the preservation of stability, asymptotic behaviour of solutions, and of periodic solutions and Hopf bifurcations. In this paper we turn to the question of oscillatory solutions.

We shall consider functional differential equations with delay or memory of the general form

$$y'(t) = \int_{-N}^{0} k(t,s,y(t-r(t,s)))ds, \quad \text{where } r \geq 0. \tag{1}$$

For a sufficiently general interpretation of the kernel k, equations of the form (1) include the general multi-delay differential equation of the form

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_n)). \tag{2}$$

If r is allowed to take negative values then (1) could be an advanced or mixed-type functional differential equation of the form

$$y'(t) = a(t)y(t) + b(t)y(t-1) + c(t)y(t+1). \tag{3}$$

[*]Corresponding author [copyright reserved]
Email addresses: njford@chester.ac.uk (Neville J Ford), y.yan@chester.ac.uk (Yubin Yan), a.malique@chester.ac.uk (Md. Abdul Malique)

However, the analysis of mixed and advanced equations goes beyond the scope of the present paper and we refer the reader, for example, to [18, 19].

Analytical results for equations of the form (2) are better developed than results for the general equation (1). However, as will become clear quite soon, most results of interest to us in this paper will be based on more general functional differential equations of the form (1) or (3).

2. Basic theory

Our interest will be in equations that are called *oscillatory*. The reader will be familiar with the concept of an oscillatory solution, but the concept of an oscillatory equation may be less familiar. For clarity we give the following definitions:

Definition 2.1. *A function u is said to be oscillatory if there exists a constant c and an increasing sequence of values $\{t_i\}$ such that*

1. $t_i \to \infty$
2. $(u(t_i) - c)(u(t_{i+1}) - c) < 0$ *for* $i = 1, 2, 3, \ldots$.

Definition 2.2. *An equation $\mathcal{E}$ is said to be oscillatory if* every *solution of $\mathcal{E}$ is an oscillatory function.*

The relationship of these ideas to differential equations is easily illustrated with reference to a second order linear ordinary differential equation with constant co-efficients of the form

$$ay''(t) + by'(t) + cy(t) = 0. \tag{4}$$

By finding the characteristic values or eigenvalues, λ_1, λ_2 (assumed here to be distinct), of the differential operator (we solve $a\lambda^2+b\lambda+c = 0$) we are able to express all solutions of (4) in the form $y(t) = Ae^{\lambda_1 t}+Be^{\lambda_2 t}$. With a moment's thought, it is clear that none of the solutions oscillate if λ_1, λ_2 are real and all the solutions oscillate if λ_1, λ_2 are complex. If one characteristic value is real and the other complex then some solutions oscillate and others do not. The term oscillatory equation is reserved for the case where *all solutions oscillate.*

This discussion applies to wider classes of equation by considering characteristic values or eigenvalues of the underlying operator.

Lemma 2.1. *Let $\mathcal{D}$ be a linear differential operator with constant coefficients such that all the characteristic values (eigenvalues) of $\mathcal{D}$ are complex then the equation $\mathcal{D}y = 0$ is oscillatory. If one or more of the characteristic values is real then the equation is non-oscillatory.*

The proof is immediate, simply by writing the solution as a linear combination of eigenfunctions and setting all but one coefficient to zero.

Remark 2.1. *The same result applies if $\mathcal{D}$ is replaced by a linear constant coefficient integral or integro-differential operator.*

The usual prototype delay differential equations for analysis are the equations

$$y'(t) = ay(t) + by(t - \tau) \tag{5}$$

and

$$y'(t) = \mu y(t - 1). \tag{6}$$

These two equations can be shown to be equivalent under an appropriate transformation.

The characteristic values for (6) are solutions to the quasi-polynomial $\lambda e^{\lambda} = \mu$ and it is easy to show that, for $\mu \in [\frac{-1}{e}, \infty)$ there is a real characteristic root while for $\mu \in (-\infty, \frac{-1}{e})$, there are no real characteristic roots.

3. Existing results

By way of illustration of the fundamental analytical ideas, consider the equation

$$y'(t) = \int_{-1}^{0} y(t - r(\varphi))dq(\varphi) \tag{7}$$

where r is a real continuous non-negative function on $[-1, 0]$ and q is a real function of bounded variation on $[-1, 0]$.

These equations generalise single and multi-term delay differential equations and various delay integro-differential equations. They are sufficiently complicated to exhibit the range of behaviours we need to study while remaining tractable to analysis. In particular the underlying linear nature of the problem makes the work amenable to analysis using characteristic values.

We have the following result:

Lemma 3.1. *Equation (7) is oscillatory if and only if $F(\lambda)$ is not equal to zero for $\lambda \in \mathbb{R}$, where*

$$F(\lambda) = \lambda - \int_{-1}^{0} e^{(-\lambda r(\varphi))}dq(\varphi). \tag{8}$$

This result can be found in the work of [17] (see also [6, 7, 21] who also develop additional criteria to identify oscillatory equations) and can be derived by the usual approach of searching for solutions of (7) that take the form $e^{\lambda t}$.

We can use these results as a basis for characterising whether or not an equation is oscillatory:

Example 3.1. *As we remarked already, the basic delay equation*

$$y'(t) = \mu y(t - 1) \tag{9}$$

is non-oscillatory for $\mu \geq \frac{-1}{e}$ and oscillatory otherwise.

Example 3.2. *A broad class of examples takes the form*

$$\begin{aligned} r(\varphi) &= \varphi^n, \text{ for some } n \in \mathbb{N} \\ q(\varphi) &= c\varphi^m, \text{ for some } c \in \mathbb{R} \text{ and some } m \in \mathbb{N}. \end{aligned}$$

When n is even, r is non-negative and we have found such problems to be non-oscillatory. When n is odd, the equation is advanced and goes beyond the scope of the present paper. Nevertheless, the approach can be extended to this case too and we shall explore this in a sequel.

Example 3.3. *If we take a sub-set of the problems given in Example 3.2 of the form*

$$\begin{aligned} r(\varphi) &= \varphi^n, \text{ for some even } n \in \mathbb{N} \\ dq(\varphi) &= r'(\varphi)d\varphi \end{aligned}$$

then we obtain a family of equations that have common analytical properties and which are all non-oscillatory.

Example 3.4. *If we consider the equation (see [21]) of the form*

$$y'(t) = \frac{-1}{a} \int_{-1}^{0} y(t - \frac{1}{5} + \theta)d\theta \tag{10}$$

for some constant $a > 0$, then the equation is oscillatory.

4. Application of numerical methods

Numerical methods to solve problems of the form (7) can be based on a simple combination of a differential equation solver and a quadrature rule (see [1, 14, 16, 20]). One could apply, for example, a linear multistep method or a Runge-Kutta method for solving the differential equation. In this paper, the numerical methods that we shall apply in our examples are examples of *linear* $\vartheta-$*methods* which generalise the Euler rule, trapezium rule and implicit Euler rule and which can be expressed either as a linear multistep method or as a Runge-Kutta method. These are convenient because they illustrate key features of both types of method and because they have simple natural quadrature rule analogues. The resulting equations take the form of a discrete Volterra equation or difference equation. We shall be able to give theoretical results that cover more general methods too.

To start with, we consider a simple constant step-size discretisation of (7). We let $N \in \mathbb{N}$, $h = \frac{1}{N} > 0$ and $y_n \approx y(nh)$ as usual and we write

$$\frac{y_{n+1} - y_n}{h} = h \sum_{j=-N}^{0} w_j \tilde{q}(j) \tilde{y}(n - \tilde{k}(j)) \tag{11}$$

Here the values w_j are the quadrature weights, $\tilde{q}$ is a weight function based on the original measure q in (7) and $\tilde{k}(j) = k(jh)$. The function $\tilde{y}$ is a dense output of the solution process. In other words $\tilde{y}(j) = y_j$ for $j \in \mathbb{N}$ and interpolates at non integer values of its argument. The interpolation required will be some combination of the values y_j at neighbouring integer-valued points. Obviously, in the case of simple multi-delay equations with constant delays, the step length may be chosen so that interpolation becomes unnecessary. The equation (11) provides an expression for y_{n+1} as a function of $y_n, y_{n-1}, \ldots, y_{n-N}$.

Of course, here (as elsewhere in this paper) we used a very simple one-step solver for the differential equation. If we choose a multistep method then we shall have a much more complicated expression, but we shall still retain the same overall idea, and we will obtain a discrete equation of the same overall form. The same observation would apply if we adopted a backward difference or central difference approach to approximating the left hand side of (7).

For a first example, consider the equation

$$y'(t) = 2 \int_{-1}^{0} y(t + \varphi) \varphi d\varphi \tag{12}$$

In other words, this is of the form of Example 3.2 with

$$q(\varphi) = \varphi^2, r(\varphi) = -\varphi$$

For a constant step size $h > 0$ and all the usual notation, the linear $\vartheta-$methods are defined in the following way: Let $\vartheta \in [0, 1]$. For the differential equation

$$y'(t) = \mathcal{F}(t, y(t)), y(0) = y_0 \tag{13}$$

the approximate solution given by the linear $\vartheta - method$ is given by

$$y_{n+1} = y_n + h(1 - \vartheta) F_n + h\vartheta F_{n+1}. \tag{14}$$

The corresponding quadrature rule approximates the integral

$$I = \int_{nh}^{(n+1)h} f(s) ds \approx h(1 - \vartheta) f(nh) + h\vartheta f((n + 1)h). \tag{15}$$

The values $\vartheta = 0, \frac{1}{2}, 1$ correspond to the familar Euler, trapezium rule and implicit Euler solvers.

Applying a discrete scheme based on an Euler rule ($\vartheta = 0$) for the differential equation and the corresponding forward rectangular rule ($\vartheta = 0$) for the quadrature we obtain

$$y_{n+1} = y_n + 2h^2 \sum_{j=0}^{N} w_j j h y_{n-j} \tag{16}$$

This is a straightforward finite order difference scheme and it can be analysed in a straightforward way.

However the following example shows how apparently simple equations may become unexpectedly complicated:

$$y'(t) = \int_{-1}^{0} y(t-\varphi^2) dq(\varphi) \text{ where } q(\varphi) = \varphi^2 \tag{17}$$

Direct application of a simple discrete scheme gives us an expression of the form

$$y_{n+1} = y_n + 2h^2 \sum_{j=0}^{N} w_j j h \tilde{y}_{n-j^2 h} \tag{18}$$

It is easy to see that now we shall need to interpolate the values of $\tilde{y}$ since we shall not always be dealing with grid-points. However there may, according to the form of the function r, be the possibility to avoid this by using a non-uniform grid for φ and thereby needing only grid values of $\tilde{y}$. This can be accomplished for the present example as we shall see later.

5. Conditions for oscillation of discrete equations

In this Section we shall discuss criteria for discrete equations, such as those that have emerged in the previous section, to be oscillatory. We focus on linear equations, where the analysis can be based on the zeros of a polynomial and we shall refer also to a characterisation for nonlinear problems.

For the linear equation

$$y_{n+1} = \sum_{j=0}^{N} a_j y_{n-j}, \tag{19}$$

by considering the characteristic equation for problems of this type, it is simple to show that the general solution may be written as a linear combination of eigenfunctions. (One needs to take account of any repeated characteristic values in the usual way.) Let the values of ψ_i be the zeros of the characteristic polynomial,

$$z^{N+1} - a_0 z^N - a_1 z^{N-1} - \ldots - a_N = 0 \tag{20}$$

then the solution takes the form $y_n = \sum_{j=1}^{N+1} b_j {\psi_j}^n$. assuming all the zeros are distinct. If zeros are repeated, a slightly more complicated expression is needed for the solution but for our purposes the conclusions will be the same.

Any particular eigenfunction ψ_i^n oscillates unless $\psi_i \in \mathbb{R}^+$ and therefore we can give the characterisation:

Lemma 5.1. *The equation (19) is oscillatory if and only if none of the zeros of (20) lie on the non-negative real axis.*

This result, based on the zeros of the characteristic polynomial, will prove most useful in our theoretical analysis, which is currently confined to linear equations. However, a more general theorem that applies also to certain nonlinear discrete problems has been given (see [5, 13]) and it may prove fruitful in the further investigation of non-linear problems.

Consider the difference equation (discrete Volterra equation),

$$y_{n+1} = y_n - \sum_{i=1}^{m} p_i f_i(y_{n-k_i}) \tag{21}$$

where $p_i > 0, k_i$ are positive integers, and f_i are continuous functions on $\mathbb{R}$.

Theorem 5.1. *Suppose that the following conditions are met:*

1. $yf_i(y) > 0 \quad$ *for* $y \neq 0, 1 \leq i \leq m$
2. $\lim_{y\to 0} \inf \frac{f_i(y)}{y} \geq 1, \quad 1 \leq i \leq m$
3. $\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1$

then every solution of (21) oscillates.

Remark 5.1. *We note that in the study of stability theory and exponential growth and decay, it is usual to linearise equations and to use the linear analysis as the basis for obtaining a close approximation to the behaviour of a non-linear problem. For the study of oscillation theory, it is clear that the situation is more complicated, and the extent to which a linear analysis provides useful insights into non-linear problems has not been established. Some examples that we have considered are non-linear, and we have seen no experimental evidence that our methods fail in these cases.*

For both continuous and discrete linear problems, the key questions relating to oscillation are determined by the location of zeros of a function. In the discrete case, the function will be a polynomial of degree that depends upon the step-size chosen for the numerical scheme, and the degree increases as the step length becomes smaller, and the approximation of the continuous problem by the discrete scheme becomes closer. Various root-finding techniques have been developed to support investigations of this type. Most have been developed as part of the investigation of stability, but some of the basic methods may be applied to the study of oscillatory behaviour too. We draw attention to the Boundary locus or D-partition method ([2]), methods of approximating the characteristic roots ([8, 22]) and approaches based on knowing the trajectory on which the roots of the characteristic equation lie (see, for example, [4, 15, 9]). For the polynomial arising from the discrete scheme, one can also use a direct polynomial solver, for example from the NAG Fortran library, or using Matlab or Mathematica. This approach is adequate for resonably low order polynomials, but becomes increasingly cumbersome as the degree of the polynomial when we employ small step lengths. The boundary locus method seems not to adapt very well to our needs, so we have developed a method based on the Principle of the Argument for estimating the number of real roots of the characteristic equation, and have adapted the theoretical treatment from [8] to provide a guarantee of the efficacy of this approach.

6. The use of the Argument Principle to count zeros

As we have seen already, the step length chosen as the basis for developing the discrete scheme determines the order of the difference system to be analysed. As has been noted in previous work (see, for example, [11, 12]) the key difficulty is that we are using a finite-dimensional approximation of the underlying infinite-dimensional continuous problem. As we reduce the step length, we shall increase the dimension of our approximation and so it will become a *better* approximation both in the sense that the smaller step length reduces local errors and also in the sense that the dimension will increase and enable a wider range of dynamical behaviours to be modelled in the approximation.

These observations show that it is likely that the *correct* behaviour of the continuous problem will only be recovered in the discrete approximation (if, indeed it is recovered at all) for quite small step lengths. This means that we shall need to find (or locate approximately) the characteristic values based on zeros of high degree polynomials.

We recall that the aim is to test the characterisation property: *The equation (19) is oscillatory if and only if none of the zeros of (20) lie on the non-negative real axis.* Thus we do not need to find the zeros of the polynomial, we merely need to count how many lie on the positive real axis or at the origin.

The Argument Principle (see, for example, [3]) provides the ideal tool for this investigation. We recall:

Theorem 6.1. *Let f be a meromorphic function on a domain D with poles $p_1, p_2, \ldots, p_m$ and zeros $z_1, z_2, \ldots, z_n$ repeated as necessary according to multiplicity. Let γ be a Jordan curve in D which does not pass through*

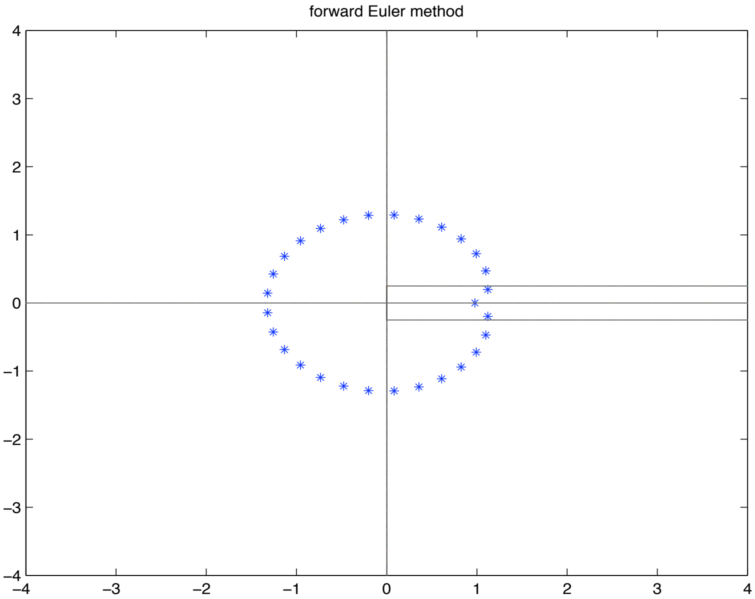

Figure 1: Application of the Principle of the Argument to count zeros of the characteristic polynomial

any of the zeros or poles, then

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = \textit{the number of zeros of } f \textit{ lying within } \gamma - \textit{ the number of poles of } f \textit{ lying within } \gamma \quad (22)$$

In the cases we are considering, f is a polynomial function and so the integral permits us to calculate the number of zeros (counting multiple zeros accordingly) lying within γ.

We let $M > 0$ be fixed and we let γ_M be the rectangle with vertices at $(0, \pm\frac{1}{M}), (M, \pm\frac{1}{M})$. We approximate, numerically, the integral on the left hand side of (22) around γ_M as we allow M to vary through a sequence of increasing values. This enables us to count the number of zeros of f lying on the positive real axis in a very straightforward way. Of course we must be careful to ensure the accuracy of the numerical method used to evaluate the integral as well as ensuring that we choose M sufficiently large to capture all the positive real zeros of f. We give examples illustrating the use of this approach in the next section. Figure 1 illustrates the method using $M = 4$ and for which the integral will calculate three zeros lying inside the rectangle. The zeros of the characteristic polynomial are marked with $*$ and as $M \to \infty$ we can see that the method will count the single real positive zero.

Table 1 records the results of the numerical calculations using different values of M to calculate the number of zeros of a polynomial lying in the rectangle. As M becomes larger, the number of zeros of the polynomial counted by the Argument Principle tends to 1. There is one positive real characteristic value for the discrete scheme. The underlying problem is non-oscillatory, as is the discrete scheme.

Step length h	M	No. of zeros by Argument Principle
0.0001	10	7
0.0001	20	1
0.0001	30	1

Table 1: Number of zeros counted by Argument Principle

7. Theoretical treatment

In this section we aim to gain a clearer understanding of the relationship between the characteristic values for the discrete scheme and those of the underlying continuous problem. It is our aim to provide a Theorem that guarantees, under appropriate conditions on the choice of scheme and step lengths, that the numerical scheme will reproduce faithfully the oscillatory characteristics of the underlying problem. We shall assume a fairly abstract approach to the discrete problem and derive a general theorem on the proximity of the characteristic roots of the discrete scheme to those of the original problem. The approach we adopt is based on the one used in [8].

We recall that the characteristic values for equation (7) satisfy:

$$F(\lambda) = \lambda - \int_{-1}^{0} e^{(-\lambda r(\varphi))} dq(\varphi) = 0 \tag{23}$$

and the condition for (7) to be oscillatory is given by Lemma 3.1.

Apply a linear multistep method of order p with generating polynomials ρ, σ to the left hand side of (7) and apply a convergent and consistent quadrature rule of order s to evaluate the integral term. In the usual notation, E represents the forward step operator and the resulting discrete equation has the form

$$\rho(E)y_n - h\sigma(E)I_n = 0 \tag{24}$$

where the quadrature weights ω_i determine the values I_n by the relation

$$I_n = h\sum_{i=1}^{\Pi} \omega_i y_{n-r_i} q_i \tag{25}$$

It follows that the characteristic polynomial for (24) can be rewritten in the form

$$\tilde{F_h}(\varphi) = \rho(\varphi)\varphi^n - h^2\sigma(\varphi)\varphi^n \sum_{i=1}^{\Pi} \omega_i \varphi^{-r_i} q_i = 0 \tag{26}$$

Now assume that φ is a characteristic root of $\tilde{F_h}$. Bearing in mind that φ represents the solution dynamics over a time-step of length $h > 0$, by comparison with the continuous case we put $\varphi = e^{\lambda t}$. It follows that

$$\frac{\rho(\varphi)}{\sigma(\varphi)} = h^2 \sum_{i=1}^{\Pi} \omega_i \varphi^{-r_i} q_i \tag{27}$$

or, using the expression for I_n,

$$\frac{\rho(e^{\lambda h})}{\sigma(e^{\lambda h})} = h\int_{-1}^{0} e^{-\lambda r(\varphi)} dq(\varphi) + \mathcal{O}(h^{s+2}) \tag{28}$$

Now we apply the following result:

Theorem 7.1 (see Theorem 2.1 in [16]). *The linear multistep method defined by the generating polynomials ρ and σ is of order $p \geq 1$ if and only if there exists $c \neq 0$ such that*

$$\rho(e^z) - z\sigma(e^z) = cz^{p+1} + \mathcal{O}(z^{p+2}), \quad z \to 0. \tag{29}$$

It follows that there exists $\mathcal{C} \neq 0$ such that

$$\frac{\rho(e^z)}{\sigma(e^z)} = z + \mathcal{C}z^{p+1} + \mathcal{O}(z^{p+2}), \quad z \to 0. \tag{30}$$

Combining this equation with (28), we obtain the following:

$$\lambda h - h\int_{-1}^{0} e^{-\lambda h r(\varphi)} dq(\varphi) = \mathcal{C}(\lambda h)^{p+1} + \mathcal{O}(\lambda h)^{p+2} + \mathcal{O}(h^{s+2}). \tag{31}$$

In other words, $F(\lambda) = \mathcal{O}(h^p) + \mathcal{O}(h^{s+1})$.

Remark 7.1. 1. *The argument is reversible and therefore we can conclude that whenever* $F(\lambda) = 0$, *it follows that* $\tilde{F}_h(e^{\lambda h}) = \mathcal{O}(h^p) + \mathcal{O}(h^{s+1})$.

2. *A similar theory can be developed, based on use of the* stability function *for Runge-Kutta methods for the differential operator instead of multistep methods.*

We are able to conclude that every root of the characteristic equation for the discrete problem corresponds to a near-zero value for the characteristic function of the continuous problem and vice versa. As we saw in [8], one cannot therefore claim that the zeros of the discrete scheme are necessarily close to zeros of the continuous problem. Instead, we considered there a sequence of step lengths $\{h_n\}$ tending to zero and the set of all characteristic zeros of the resulting sequence of discrete problems. From the correspondence $\varphi = e^{\lambda h}$ (which sets up the correspondence between real characteristic value for the continuous problem and non-negative characteristic values for the discrete cheme) the cluster points of the values of λ as h varies provide approximations to the characteristic values of the continuous problem. We can summarise in the following

Theorem 7.2. *Let (7) be oscillatory. For a fixed step length* $h = \frac{1}{N}$, *apply a convergent and consistent quadrature rule for the integral and a convergent and consistent linear multi-step formula for the differential equation. In the limit as* $N \to \infty$ *the resulting discrete scheme will be oscillatory.*

Let (7) be non-oscillatory. For a fixed step length $h = \frac{1}{N}$, *apply a convergent and consistent quadrature rule for the integral and a convergent and consistent linear multi-step formula for the differential equation. In the limit as* $N \to \infty$ *the resulting discrete scheme will be non-oscillatory.*

8. Examples

8.1. Example 1

Consider the non-oscillatory problem

$$y'(t) = 3y(t-1).$$

Applying the forward Euler rule where $h > 0, Nh = 1, t_n = nh, y_n \approx y(nh)$ we obtain

$$y_{n+1} - y_n - 3hy_{n-N} = 0.$$

The characteristic equation can be written

$$\theta^{N+1} - \theta^N - 3h\theta^{N-N} = 0.$$

Figure 2 represents the characteristic values for $h = 0.01$ and Table 2 shows that the discrete problem is non-oscillatory.

Step length	M	No. of zeros
0.01	2	17
0.001	10	3
0.001	20	3
0.001	30	1
0.001	40	1
0.0001	40	1

Table 2: Example 1: Number of positive real zeros by Argument principle

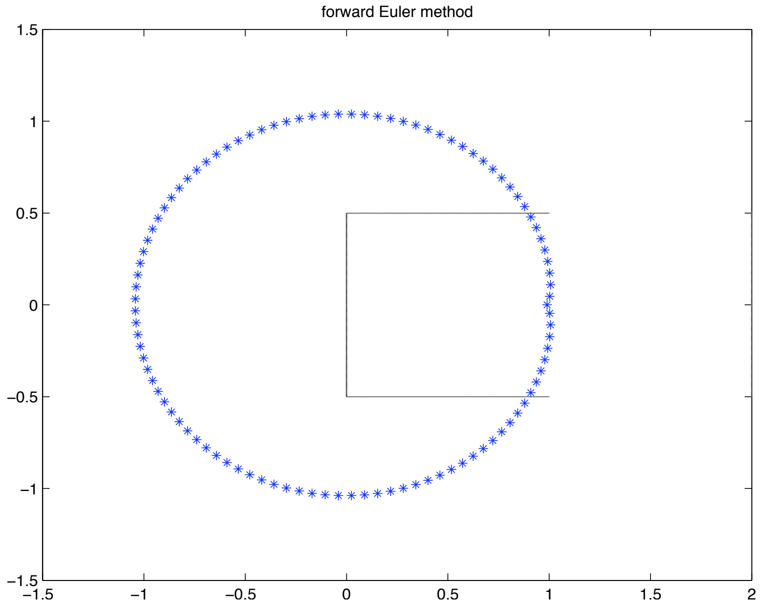

Figure 2: Example 1: Characteristic plot for h=0.01, M=2

8.2. Example 2

For the oscillatory equation

$$y'(t) = -y(t-1).$$

Applying the forward Euler rule with the usual notation, we obtain

$$y_{n+1} - y_n + hy_{n-N} = 0.$$

The characteristic equation is

$$\theta^{N+1} - \theta^N + h\theta^{N-N} = 0$$

and the characteristic roots are represented in Figure 3 and Table 3, confirming that the problem is oscillatory.

step length	M	No. of zeros
0.01	2	18.0955
0.001	10	4
0.001	20	2
0.001	30	2
0.001	40	2
0.0001	40	2
0.0001	large	0

Table 3: Example 2: Number of positive real roots by Argument principle

8.3. Example 3

Let us consider the non-oscillatory FDE

$$y'(t) = 2\int_{-1}^{0} (t - \varphi^2)\varphi d\varphi.$$

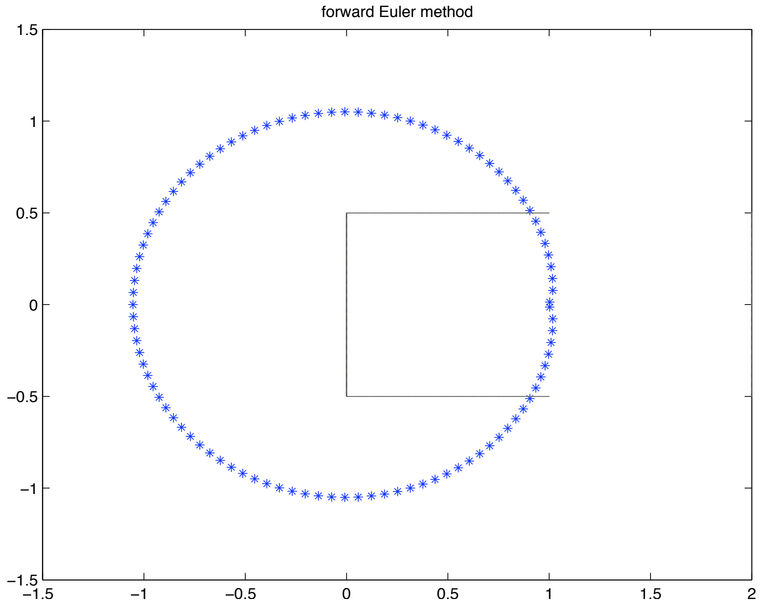

Figure 3: Example 2: Characteristic plot for h=0.01,M=2

Applying the forward Euler rule for the differential equation, and a specially selected quadrature rule with step length $\sqrt{h}$, we obtain the discrete scheme,

$$\frac{y_{n+1} - y_n}{h} = 2 \sum_{k=-N}^{0} w_k y_{n-k^2}.\sqrt{h}k.\sqrt{h}$$

where $h > 0, t = nh, \varphi = \sqrt{h}k, y(nh) \approx y_n, w_0 = w_{-N} = \frac{1}{2}, w_k = 1, Nh = 1.$

$$y_{n+1} - y_n = 2h^2 \sum_{k=-N}^{0} k w_k y_{n-k^2}$$

and the characteristic equation is

$$\theta^{N^2+1} - \theta^{N^2} + \frac{2}{N^2} w_1 \theta^{N^2-1^2} + ... + \frac{2}{N^2} N w_{-N} \theta^{N^2-N^2} = 0.$$

Figure 4 and Table 4 confirm that the discrete problem is non-oscillatory.

Step length	M	Number of zeros
0.01	2	13.0482
0.001	10	3.0002
0.001	20	1
0.001	30	1
0.001	40	1
0.0001	40	1

Table 4: Example 3: Number of positive real roots by Argument principle

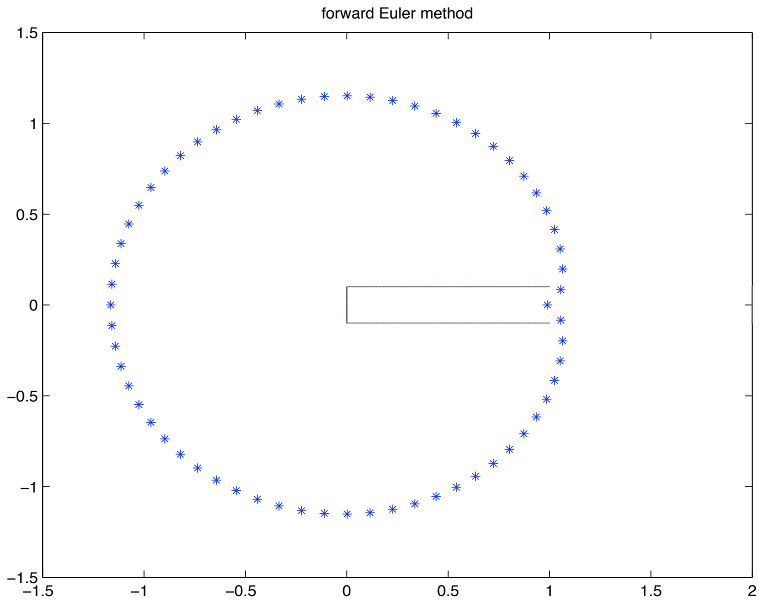

Figure 4: Example 3: Characteristic plot for for h=0.0125, M=10

8.4. Example 4

We consider the oscillatory equation:

$$y'(t) = -\frac{1}{a}\int_{-1}^{0} y(t-\frac{1}{5}+\theta)d\theta, \quad a>0 \tag{32}$$

Applying the Euler forward rule, with t=nh, $y_n \approx y(nh)$, $\theta = jh$, $Nh = \frac{1}{5}$, $0 < a < 2$ we obtain:

$$\frac{y_{n+1}-y_n}{h} = -\frac{1}{a}\sum_{j=-5N}^{0} y_{n-N+j}.h. \tag{33}$$

and the characteristic equation takes the form

$$\frac{h^2}{a}[\theta^0+\theta^1+\theta^2+......+\theta^{5N}] - \theta^{6N} + \theta^{6N+1} = 0 \tag{34}$$

Figure 5 and Table 5 confirm that the discrete problem is also oscillatory.

Step length h	M	Number of zeros
0.01	2	22
0.01	4	10
0.01	10	4
0.01	20	2
0.01	30	2
0.01	40	2
0.01	large	0

Table 5: Example 4: Number of positive real roots by Argument principle

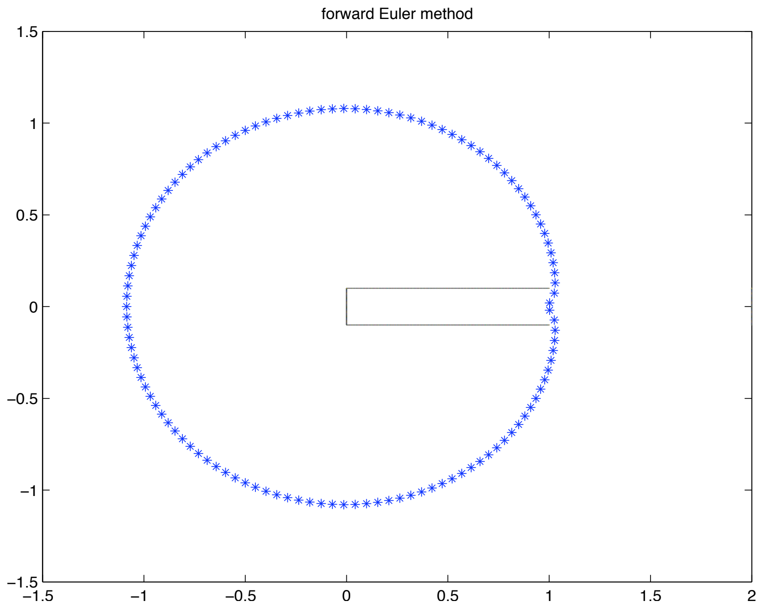

Figure 5: Example 4: Characteristic plot for h=0.01, M=20

9. Conclusions

As we have seen, the numerical approach introduced here does provide a reliable method for determining whether or not linear functional differential equations are oscillatory. Based on the experiments we have tried, the technique works also for non-linear problems, but there is a need for further analytical results in this case.

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